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Morita equivalences between blocks of finite group algebras

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1. Introduction and notation

In representation theory of finite groups, particularly, in modular representation theory, studying structure of p-blocks (block algebras) of finite groups G, where p is a prime number, is one of the most important and interesting things.

Notation 1.1. Throughout this note we use the following notation and terminology. We denote by G always a finite group, and let p be a prime. Then, a triple $(\mathcal{K}, \mathcal{O}, k)$ is so-called a p-modular system, which is big enough for all finitely many finite groups which we are looking at, including G. Namely, $\mathcal{O}$ is a complete descrete valuation ring, $\mathcal{K}$ is the quotient field of $\mathcal{O}$, $\mathcal{K}$ and $\mathcal{O}$ have characteristic zero, and k is the residue field $\mathcal{O}/\mathrm{rad}(\mathcal{O})$ of $\mathcal{O}$ such that k has characteristic p. We mean by "big enough" above that $\mathcal{K}$ and k are both splitting fields for the finite groups mentioned above. Let A be a block of $\mathcal{O}G$ (and sometimes of kG) with a defect group P. We denote by mod-kG and by mod-A the categories of finitely generated right kG- and A-modules, respectively. We write $B_0(kG)$ for the principal block algebra of kG. For the notation and terminology we shall not explain precisely, see the books of [2] and [3].

Setup 1.2. Throughout this note all the time except in Theorem **2.1** our situation is the following: Namely, G and H are finite groups which have the same Sylow p-subgroup P, and hence $P \subseteq G \cap H$. Assume that $\tilde{G}$ is a normal subgroup of G and $\tilde{H}$ is a normal subgroup of H such that $\tilde{G}$ and $\tilde{H}$ have the same Sylow p-subgroup $\tilde{P}$, and hence $\tilde{P} \subseteq \tilde{G} \cap \tilde{H}$, and moreover that $G/\tilde{G} \cong H/\tilde{H}$.

Remark 1.3. If the factor group $G/\tilde{G}$ is p'-groups, then we know essentially by the famous result due to H.Maschke (1898) that the ring extension $k\tilde{G} \subseteq kG$ is a so-called *separable* extension. Then, roughly speaking, mod-kG and mod-$k\tilde{G}$ are in some sense *similar* (of course, even the numbers of simples in the two module categories are different, though). Therefore, much more interesting situation should be the case where $|G/\tilde{G}|$ is divisible by p. Then, here comes our situation.

Our situation 1.4. We still keep the setup **1.2**. In addition we assume that the factor groups $G/\tilde{G} \cong H/\tilde{H}$ are p-groups. Surely, the factor groups are isomorphic to $P/\tilde{P}$, too. Then, we naturally come to the following questions.

Questions 1.5. Our main concern in this note is the following:

(i) If there is a *nice* equivalence between mod-$k\tilde{G}$ and mod-$k\tilde{H}$, can we lift it to a *nice* equivalence between mod-kG and mod-kH?

(ii) If there is a *nice* equivalence between mod-kG and mod-kH, can we descend it to a *nice* equivalence between mod-$k\tilde{G}$ and mod-$k\tilde{H}$?

2. Results

In this short section we shall list two results which come up from Question **1.5**.

Theorem 2.1. *Assume* **1.4**, *however, note that we do not assume that P and $\tilde{P}$ are Sylow p-subgroups of G and $\tilde{G}$, respectively. Namely, P is just a p-subgroup of G and also of H, and $\tilde{P}$ is just a p-subgroup of $\tilde{G}$ and also of $\tilde{H}$ We assume then that P is a defect group of A and B, and $\tilde{P}$ is a defect group of $\tilde{A}$ and $\tilde{B}$. Moreover, we suppose* *that the factor groups* $Q := G/\tilde{G} \cong H/\tilde{H} \cong P/\tilde{P}$ *are just cyclic group* C_p *of order* p, *and that* A, $\tilde{A}$, B, $\tilde{B}$ *respectively are block algebras of* kG, $k\tilde{G}$, kH, $k\tilde{H}$, *such that* A *covers* $\tilde{A}$ *and* B *covers* $\tilde{B}$. *Set* $\Delta Q := \{(u,u) \in Q \times Q | u \in Q\}$. *We assume, in addition, that* $\tilde{A}$ *and* $\tilde{B}$ *are both* ΔQ*-invariant, that is, they are stable under conjugation action by all elements in* Q. *Set furthermore that* $\Delta := (\tilde{G} \times \tilde{H})\Delta Q = (\tilde{G} \times \tilde{H})\Delta P = (\tilde{G} \times \tilde{H})\Delta G = (\tilde{G} \times \tilde{H})\Delta H$. *Then, we get the following: Suppose that there is a bounded complex* $\tilde{M}^\bullet \in C^{\mathrm{b}}(\mathcal{O}\tilde{A}\text{-mod-}\mathcal{O}\tilde{B})$ *of finitely generated* $(\mathcal{O}\tilde{A}, \mathcal{O}\tilde{B})$*-bimodules such that*

(1) $\tilde{M}^\bullet \otimes_{\mathcal{O}} \mathcal{K}$ *induces an isometry* $\tilde{I}$ *from* $\mathbb{Z}\mathrm{Irr}(\tilde{A})$ *to* $\mathbb{Z}\mathrm{Irr}(\tilde{B})$;
(2) $\tilde{M}^\bullet$ *is perfect (exact), that is, all terms in the complex* $\tilde{M}^\bullet$ *are projective as left* $\mathcal{O}\tilde{G}$*-modules and also as right* $\mathcal{O}\tilde{H}$*-modules (and hence the isometry* $\tilde{I}$ *above is perfect),*
(3) *the complex* $\tilde{M}^\bullet$ *extends from* $\tilde{G} \times \tilde{H}$ *to* Δ.

Then, we can define a bounded complex $M^\bullet := \tilde{M}^\bullet_{\tilde{G}\times\tilde{H}\to\Delta} \uparrow^{G\times H} \in C^{\mathrm{b}}(\mathcal{O}A\text{-mod-}\mathcal{O}B)$, *and the new complex* $M^\bullet$ *induces a perfect isometry from* $\mathbb{Z}\mathrm{Irr}(A)$ *to* $\mathbb{Z}\mathrm{Irr}(B)$. *where* $M^\bullet := \tilde{M}^\bullet_{\tilde{G}\times\tilde{H}\to\Delta} \uparrow^{G\times H}$ *is an induced complex by applying the functor* $- \otimes_{\mathcal{O}\Delta} \mathcal{O}[G \times H]$ *to the bounded complex* $\tilde{M}^\bullet$.

Corollary 2.2. *We easily get* [1, Example 4.3] *in our previous paper by making use of Theorem* **2.1**.

Theorem 2.2. *Assume* **1.4**. *Here we assume that* P *is a Sylow* p*-subgroup of* G *and* H, *and also* $\tilde{P}$ *is a Sylow* p*-subgroup of* $\tilde{G}$ *and* $\tilde{H}$. *Moreover, we suppose that the factor groups* $Q := G/\tilde{G} \cong H/\tilde{H} \cong P/\tilde{P}$ *are isomorphic finite* p*-groups, and that* A, $\tilde{A}$, B, $\tilde{B}$ *respectively are* **principal** *block algebras of* kG, $k\tilde{G}$, kH, $k\tilde{H}$ *Set* $\Delta P := \{(u,u) \in P \times P | u \in P\}$. *Moreover, we denote by* $\mathrm{Scott}(G \times H, \Delta P)$ *the (Alperin-)Scott module in* $G \times H$ *with respect to a subgroup* ΔP *of* $G \times H$, *see* [2, Chap.4 Theorem 8.4, Corollary 8.5]. *Then, we get the following: If* ${}_AM_B := \mathrm{Scott}(G \times H, \Delta P)$ *induces a Morita equivalence (and hence it is a Puig equivalence) between* A *and* B, *then* ${}_{\tilde{A}}\tilde{M}_{\tilde{B}} := \mathrm{Scott}(\tilde{G} \times$

$\tilde{H}, \Delta\tilde{P})$ induces a Morita equivalence (and hence it is a Puig equivalence) between $\tilde{A}$ and $\tilde{B}$. (Recall that $A := B_0(kG) = \mathrm{Scott}(G\times G, \Delta\tilde{P})$ and $B := B_0(kH) = \mathrm{Scott}(H \times H, \Delta\tilde{P})$.

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